Topics Covered

(NB: This list is not intended to be a complete list...)

Solving linear systems

↳ Gauss-Jordan Elimination

↳ Row reduction and RREF matrices

Geometry and linear systems

↳ dot product / angle formula.

Matrices and Matrix operations

↳ addition, scalar multiplication, matrix product, transpose

Vector spaces

↳ subspaces and subspace test

$S \subseteq V$ w/ $S \ni \vec{0}$ and $ax + by \in S$ for all scalar a, b and all $x, y \in S$. $\Rightarrow S \leq V$

↳ span and linear independence

↳ $S \subseteq V$ is lin. ind. when $\sum c_i s_i = 0 \Rightarrow c_i = 0$ for all i

↳* Bases and dimension

Linear maps

↳ linearity condition

↳ kernel and range spaces ($\approx$ null and column spaces)

↳ injectivity and surjectivity.

L is inj iff $\ker(L) = 0$

↳ Matrix representation

↳ Rank-Nullity Theorem ⇝ $\mathrm{rank}(L) + \mathrm{nullity}(L) = \dim(\mathrm{dom}(L))$

↳ Linear operators * $L : V \to V$

More on Matrices

↳ determinant

↳ elementary matrices *

↳ inversion of matrices

* Change of Basis

Eigenspaces

↳ Characteristic polynomial

↳ eigenvalues and eigenvectors

↳ Complex vector spaces

Diagonalization of matrices / linear operators

↳ Similar matrices ⇝ $B = PAP^{-1}$

↳ diagonalizability ⇝ $M = PDP^{-1}$

Orthogonality (in $\mathbb{R}^n$).

↳ orthogonal projection

*↳ orthogonal complement $\boxed{\mathrm{Col}(M)^{\perp} = \mathrm{null}(M^T)}$

↳ Gram-Schmidt process *

↳ orthogonal matrices $A^{-1} = A^T$ (i.e. $A^T A = I$)

Symmetric Matrices ⇝ $A^T = A$

*↳ Transpose ⇝ $(AB)^T = B^T A^T$, $(A+B)^T = A^T + B^T$ ...

*↳ Real symmetric matrices have all eigenvalues real.

↳ Orthogonal diagonalizability $M = QDQ^T$ for Q orthogonal, D diagonal.

↳ $\boxed{\begin{array}{l}M \text{ symmetric iff} \\ M \text{ ortho. diag'able.}\end{array}}$

ker(L) = kernel of a linear map

$= \{ v \in \mathrm{dom}(L) : L(v) = 0 \}$

---

null(M) = null space of matrix M

= Solution set to $M\vec{x} = 0$

$= \ker(L_M)$ where $\mathrm{Rep}_{\mathcal{E}_n, \mathcal{E}_m}(L_M) = M$.

Point: kernel is associated to a linear map, whereas null space is associated to a matrix.

↳ often to compute a kernel of a linear map, we first compute the null space of an associated matrix, and then we convert that back into a kernel

Ex: The linear map $L : \mathcal{P}_3(\mathbb{R}) \to \mathbb{R}^3$ given by

$$L(a_0 + a_1 x + a_2 x^2 + a_3 x^3) = \begin{pmatrix} a_0 + a_1 \\ a_1 + a_3 \\ a_0 + a_3 \end{pmatrix}.$$

to compute ker(L), we will compute null space of an associated matrix. Let $\mathcal{B} = \{1, x, x^2, x^3\} \subseteq \mathcal{P}_3(\mathbb{R})$.

w.r.t. $\mathcal{B}$, L is represented by:

$$\left[[L(1)]_{\mathcal{E}_3} \,\middle|\, [L(x)]_{\mathcal{E}_3} \,\middle|\, [L(x^2)]_{\mathcal{E}_3} \,\middle|\, [L(x^3)]_{\mathcal{E}_3} \right].$$

$$= \begin{array}{cccc} a_0 & a_1 & a_2 & a_3 \end{array}$$
$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} = M$$

$$\mathrm{null}(M) = \mathrm{null}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \mathrm{null}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{bmatrix} = \cdots = \mathrm{null}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= \text{null}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{cases} a_0 = 0 \\ a_1 = 0 \\ a_2 = t \\ a_3 = 0 \end{cases} \leftarrow$$

$$\therefore \ker(L) = \{ v \in P_3(\mathbb{R}) : a_0 + a_1 x + a_2 x^2 + a_3 x^3 = v, \ a_0 = 0,\ a_1 = 0,\ a_2 = t \in \mathbb{R},\ a_3 = 0 \}$$

$$= \{ t x^2 : t \in \mathbb{R} \} \leq P_3(\mathbb{R}). \leftarrow$$

indicates

by contrast: $\text{null}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \left\{ \begin{bmatrix} 0 \\ 0 \\ t \\ 0 \end{bmatrix} \in \mathbb{R}^4 : t \in \mathbb{R} \right\} \leq \mathbb{R}^4$

M an $m \times n$ matrix can represent a linear map from an n-dimensional space to an m-dimensional space.

Ex: Let $M = \begin{bmatrix} 2 & 2 & 4 \\ 1 & 0 & 1 \\ 3 & 0 & 2 \end{bmatrix}$

↳ Rmk: $\text{null}(M) = \ker(L_M)$ where $L_M : \mathbb{R}^n \to \mathbb{R}^m$ has $L_M(x) = Mx$.
and $\text{col}(M) = \text{ran}(L_M)$ (same L_M as above)

Sol: To compute those, we compute RREF(M) because:

null(M) = null(RREF(M)) AND

col(M) has basis the columns of M corresp to leading 1's in RREF(M).

$$\begin{bmatrix} 2 & 2 & 4 \\ 1 & 0 & 1 \\ 3 & 0 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 4 \\ 3 & 0 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ null(M) = null(I_3) has $\begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases}$ so null$(M) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

has basis $\boxed{\emptyset}$. $col(M)$ has basis $\left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} \right\}$.

Can't be simplified... row operations change column spaces...

Remark: $\text{nullity}(L_M) = 0 = \dim(\text{null}(M))$

$\text{rank}(L_M) = 3 = \dim(\text{col}(M))$

so $\dim(\mathbb{R}^3) = 3 = 0 + 3 = \text{nullity}(L_M) + \text{rank}(L_M)$

Ex: $\begin{bmatrix} 1 & 0 & 1 \\ -1 & 5 & -2 \\ 0 & 5 & -1 \end{bmatrix} \leftarrow$

$\text{nullity}(L_M) = 1$, $\text{rank}(L_M) = 2$. (check...)

L is <u>injective</u> when for all $x, y \in \text{dom}(L)$ we have $L(x) = L(y)$ implies $x = y$. ✱

"distinct inputs map to distinct outputs"

linear $L : V \to W$ is injective if and only if $\underline{\ker(L) = 0}$.

L is <u>surjective</u> when for all $y \in \text{cod}(L)$ there is an $x \in \text{dom}(L)$ such that $L(x) = y$.

"every element of the codomain is an output".

<u>Rank-Nullity Thm</u>: $\underline{\text{rank}(L)} + \underline{\text{nullity}(L) = \dim(\text{dom}(L))}$.

if $\text{rank}(L) = \dim(\text{cod}(L))$, then L is surjective.

L is <u>bijective</u> when it is both surjective and injective.

$\hookrightarrow$ Linear L is bijective iff L is an isomorphism...